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Operator matrices for describing guiding propagation in circular bianisotropic fibres

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Abstract

Operator techniques are developed for describing electromagnetic waves in circular waveguides, on the basis of which an evolution operator (characteristic matrix) and impedance tensors of a homogeneous circular bianisotropic layer are determined. We investigate solutions of Maxwell's equations depending on the dielectric permittivity and magnetic permeability tensors and the gyration pseudotensors. We show that the solutions are expressed in terms of the Bessel functions with a matrix argument for the media isotropic in the fibre cross-section. The method proposed is applied for an analytical derivation of dispersion equations of isotropic and bianisotropic waveguides.

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1. Introduction

Guiding propagation of electromagnetic waves has been investigated in detail, for example, in [1–13], where a number of methods for mode analysis of planar waveguides and circular fibres are worked out. The radiation in a planar waveguide can be described using ray analysis [4, 7] or wave analysis (for instance, matrix approach) [5, 10]. For bianisotropic [14, 15] waveguides the matrix approach is most convenient, because it allows us to determine an algorithm of the problem, which does not depend on the complexity of a medium (the matrices are cumbersome, but well known). That is why one can write and numerically solve a dispersion equation for any multilayer waveguide.

Operator matrices were introduced by Halmos in [16] as a correspondence between operators and matrices induced by the direct expansion of a space. In electrodynamics and optics, the operator matrices are considered in the space generated by the direct sum of the two three-dimensional spaces and are expressed by the block matrices [17, 18]. The matrix blocks are the tensors of the three-dimensional space for operating with which we use methods of the coordinate-free tensor algebra [14, 19, 20]. For a bianisotropic stratified medium, the operator matrices can be represented in the form of 4×4 matrices [21] or 6×6 matrices [10] (in that case the matrices are degenerate, and instead of the matrix inversion we should use pseudoinversion [18]). Matrix formalism is based on the use of four field components of six: two electric and two magnetic field components. These field components satisfy a system of differential equations of the first order and lie in the plane of a film being continuous on this plain interface. Dispersion equations of a bianisotropic waveguide are written by means of an evolution operator (characteristic matrix) of a film and impedance tensors of a substrate and a cover. The evolution operator relates the fields in two different points—on the film interfaces. In that way one can investigate any multilayer waveguide, i.e. find modes and their polarizations, energy flux in a film. The operator techniques have also been applied for determining the reflection and transmission coefficients of a stratified bianisotropic media [10, 22, 23].

In this paper, we present the operator method for describing electromagnetic waves in multilayer bianisotropic fibres. Multilayer waveguides play an important part in the fibre optics. They increase a domain of the single mode propagation and localize energy in the core to a greater extent than two-layer fibres. Therefore, the core of a multilayer fibre can be made of the larger diameter, than the core of a two-layer fibre, and requests for their mating decrease. In addition, multilayer fibres have such dispersion characteristics that they can be used for the dispersion compensation [6, 24].

Besides the introduction and conclusion, the paper contains four sections. All results are obtained for medium tensors which can be decomposed into dyads with radial-dependant coefficients in cylindrical coordinates. In section 2, we derive the system of differential equations of the first order for four tangential field components. These components lie in the plane tangent to the surface of a circular cylinder and are continuous on the boundary. Section 3 is devoted to determining dispersion equations, mode polarizations and energy localization coefficient for a multilayer bianisotropic fibre with known evolution operators and impedance tensors of each circular layer. In section 4, the general solution of obtained equations is given for a homogeneous medium. It is expressed by a power series of the radial coordinate. The general formulae for evolution operators and impedance tensors are presented. In section 5 we determine the types of media, for which solutions of the Maxwell equations are simplified significantly. In particular, the solutions are expressed in terms of the Bessel functions with tensor (matrix) argument for bianisotropic media isotropic in the fibre cross-section. Finally, we consider the examples of isotropic and bianisotropic fibres.

2. Operator matrices for bianisotropic fibre optics

Let us consider a monochromatic electromagnetic wave of frequency ω in a circular bianisotropic waveguide. For describing electromagnetic fields, we use the basis vectors of the cylindrical coordinates (r, φ, z): $\boldsymbol{e}_r(\varphi)$, $\boldsymbol{e}_\varphi(\varphi)$ and $\boldsymbol{e}_z$. Axis z determines the guiding direction, and for the unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_z$ we have the following relationships:

$$\boldsymbol{e}_r \times \boldsymbol{e}_\varphi = \boldsymbol{e}_z \qquad \frac{\partial \boldsymbol{e}_\varphi}{\partial \varphi} = -\boldsymbol{e}_r \qquad \frac{\partial \boldsymbol{e}_r}{\partial \varphi} = \boldsymbol{e}_\varphi. \tag{1}$$

The Maxwell equations in the cylindrical coordinates take the form

$$\begin{aligned}\left(\boldsymbol{e}_z^\times \frac{\partial}{\partial z} + \boldsymbol{e}_r^\times \frac{\partial}{\partial r} + \boldsymbol{e}_\varphi^\times \frac{1}{r}\frac{\partial}{\partial \varphi}\right) \boldsymbol{H}(\boldsymbol{r}, t) &= \frac{1}{c}\frac{\partial \boldsymbol{D}(\boldsymbol{r}, t)}{\partial t} \\ \left(\boldsymbol{e}_z^\times \frac{\partial}{\partial z} + \boldsymbol{e}_r^\times \frac{\partial}{\partial r} + \boldsymbol{e}_\varphi^\times \frac{1}{r}\frac{\partial}{\partial \varphi}\right) \boldsymbol{E}(\boldsymbol{r}, t) &= -\frac{1}{c}\frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t}\end{aligned} \tag{2}$$

where $\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{B}$ and $\boldsymbol{D}$ are the complex vectors of strength and induction of the magnetic and electric fields, $\boldsymbol{b}^{\times}$ is the antisymmetric second rank tensor dual of an arbitrary vector $\boldsymbol{b}$ in the three-dimensional space $\mathcal{U}$ ($(\boldsymbol{b}^{\times})_{ik} = e_{ijk}b_j$, e_{ijk} is the Levi-Civita pseudotensor). Here and further we use covariant coordinate-free formalism for describing electromagnetic waves in complex (anisotropic, biisotropic, bigyrotropic, bianisotropic) media [14, 17]. The constitutive equations relate the Fourier images of electromagnetic field vectors and are written for bianisotropic media in the form

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{r}, \omega) &= \varepsilon(\boldsymbol{r}, \omega)\boldsymbol{E}(\boldsymbol{r}, \omega) + \alpha(\boldsymbol{r}, \omega)\boldsymbol{H}(\boldsymbol{r}, \omega) \\ \boldsymbol{B}(\boldsymbol{r}, \omega) &= \kappa(\boldsymbol{r}, \omega)\boldsymbol{E}(\boldsymbol{r}, \omega) + \mu(\boldsymbol{r}, \omega)\boldsymbol{H}(\boldsymbol{r}, \omega) \end{aligned} \tag{3}$$

where ε and μ are the dielectric permittivity and magnetic permeability tensors, α and κ are the gyration pseudotensors. In this investigation we consider tensors $\varepsilon, \mu, \alpha, \kappa$, which can be presented as a decomposition of dyads with coefficients depending on the radial coordinate r:

$$\xi(r, \varphi) = \sum_{i,j=1}^{3} \xi_{ij}(r)\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \begin{pmatrix} \xi_{rr} & \xi_{r\varphi} & \xi_{rz} \\ \xi_{\varphi r} & \xi_{\varphi\varphi} & \xi_{\varphi z} \\ \xi_{zr} & \xi_{z\varphi} & \xi_{zz} \end{pmatrix} \qquad \xi \in \{\varepsilon, \mu, \alpha, \kappa\} \tag{4}$$

where $\boldsymbol{e}_1 = \boldsymbol{e}_r, \boldsymbol{e}_2 = \boldsymbol{e}_\varphi, \boldsymbol{e}_3 = \boldsymbol{e}_z$, $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ is a dyad. Using the Fourier transformation, the fields can be written as

$$\begin{pmatrix} \boldsymbol{H}(\boldsymbol{r}, t) \\ \boldsymbol{E}(\boldsymbol{r}, t) \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\omega t} \int_R \mathrm{d}\beta\, \mathrm{e}^{\mathrm{i}\beta z} \sum_{\nu \in Z} \mathrm{e}^{\mathrm{i}\nu\varphi} \begin{pmatrix} \boldsymbol{H}(r, \varphi, \beta, \nu) \\ \boldsymbol{E}(r, \varphi, \beta, \nu) \end{pmatrix} \tag{5}$$

where β in the argument of the Fourier images is sometimes called the mode propagation constant. Since the tensors $\varepsilon, \mu, \alpha, \kappa$ are of form (4), we can search the Fourier images $\boldsymbol{H} = \boldsymbol{H}(r, \varphi, \beta, \nu)$ and $\boldsymbol{E} = \boldsymbol{E}(r, \varphi, \beta, \nu)$ as a basis vector decomposition with the radial-dependant coordinates (i.e. $\boldsymbol{H} = H_r(r)\boldsymbol{e}_r(\varphi) + H_\varphi(r)\boldsymbol{e}_\varphi(\varphi) + H_z(r)\boldsymbol{e}_z$). Then we easily calculate

$$\frac{\partial(\boldsymbol{H}(r, \varphi, \beta, \nu)\, \mathrm{e}^{\mathrm{i}\nu\varphi})}{\partial\varphi} = \mathrm{e}^{\mathrm{i}\nu\varphi}\left(\mathrm{i}\nu\boldsymbol{H} + H_\varphi \frac{\partial \boldsymbol{e}_\varphi}{\partial\varphi} + H_r \frac{\partial \boldsymbol{e}_r}{\partial\varphi}\right) = \mathrm{e}^{\mathrm{i}\nu\varphi}\left(\mathrm{i}\nu + \boldsymbol{e}_z^{\times}\right)\boldsymbol{H}(r, \varphi, \beta, \nu). \tag{6}$$

By substituting (5) and (6) into Maxwell's equations (2) we obtain

$$\begin{aligned} \boldsymbol{e}_r^{\times} \frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}r} + \left(\mathrm{i}\beta\boldsymbol{e}_z^{\times} + \frac{\mathrm{i}\nu}{r}\boldsymbol{e}_\varphi^{\times} + \frac{1}{r}\boldsymbol{e}_z \otimes \boldsymbol{e}_\varphi\right)\boldsymbol{H} &= -\mathrm{i}k(\varepsilon\boldsymbol{E} + \alpha\boldsymbol{H}) \\ \boldsymbol{e}_r^{\times} \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}r} + \left(\mathrm{i}\beta\boldsymbol{e}_z^{\times} + \frac{\mathrm{i}\nu}{r}\boldsymbol{e}_\varphi^{\times} + \frac{1}{r}\boldsymbol{e}_z \otimes \boldsymbol{e}_\varphi\right)\boldsymbol{E} &= \mathrm{i}k(\mu\boldsymbol{H} + \kappa\boldsymbol{E}) \end{aligned} \tag{7}$$

where $k = \omega/c$ is the vacuum wave number. In equations (7) there is the one r-derivative. The basis vectors $\boldsymbol{e}_r, \boldsymbol{e}_\varphi, \boldsymbol{e}_z$ determine a vector structure of the fields $\boldsymbol{H}$ and $\boldsymbol{E}$. Equations (7) in coordinate representation do not depend on the angle coordinate φ, therefore, a basis vector decomposition of the Fourier images actually holds true.

From equations (7), one can write four differential equations of the first order and two algebraic equations. By means of these algebraic equations we can rewrite (7) for new variables—tangential components of the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields. The tangential components lie in the plane tangent to the surface of a circular cylinder and are continuous on the cylindrical boundary. Thus, the tangential components are of the form $\boldsymbol{E}_\mathrm{t} = I\boldsymbol{E}, \boldsymbol{H}_\mathrm{t} = I\boldsymbol{H}$, where $I = 1 - \boldsymbol{e}_r \otimes \boldsymbol{e}_r = -\boldsymbol{e}_r^{\times 2}$ is the projection operator onto the plane normal to the vector $\boldsymbol{e}_r$. Introducing vector $\boldsymbol{u} = (\beta/k)\boldsymbol{e}_\varphi - \nu/(kr)\boldsymbol{e}_z$ and taking into consideration relationships $\boldsymbol{u}\boldsymbol{H} = \boldsymbol{e}_r\varepsilon\boldsymbol{E} + \boldsymbol{e}_r\alpha\boldsymbol{H}$, $\mathbf{uE} = -\boldsymbol{e}_r\mu\boldsymbol{H} - \boldsymbol{e}_r\kappa\boldsymbol{E}$, we obtain a connection between the complete field vectors and their tangential components

$$\begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{E} \end{pmatrix} = V \begin{pmatrix} \boldsymbol{H}_\mathrm{t} \\ \boldsymbol{E}_\mathrm{t} \end{pmatrix} \qquad V = \begin{pmatrix} I + \boldsymbol{e}_r \otimes \boldsymbol{v}_1 & \boldsymbol{e}_r \otimes \boldsymbol{v}_2 \\ \boldsymbol{e}_r \otimes \boldsymbol{v}_3 & I + \boldsymbol{e}_r \otimes \boldsymbol{v}_4 \end{pmatrix} \tag{8}$$

where

$$\boldsymbol{v}_1 = \delta_r(\kappa_{rr}\boldsymbol{e}_r\alpha I - \varepsilon_{rr}\boldsymbol{e}_r\mu I - \kappa_{rr}\boldsymbol{u}) \qquad \boldsymbol{v}_2 = \delta_r(\kappa_{rr}\boldsymbol{e}_r\varepsilon I - \varepsilon_{rr}\boldsymbol{e}_r\kappa I - \varepsilon_{rr}\boldsymbol{u})$$
$$\boldsymbol{v}_3 = \delta_r(\alpha_{rr}\boldsymbol{e}_r\mu I - \mu_{rr}\boldsymbol{e}_r\alpha I + \mu_{rr}\boldsymbol{u}) \qquad \boldsymbol{v}_4 = \delta_r(\alpha_{rr}\boldsymbol{e}_r\kappa I - \mu_{rr}\boldsymbol{e}_r\varepsilon I + \alpha_{rr}\boldsymbol{u}) \tag{9}$$
$$\delta_r = (\varepsilon_{rr}\mu_{rr} - \alpha_{rr}\kappa_{rr})^{-1} \qquad \xi_{rr} = \boldsymbol{e}_r\xi\boldsymbol{e}_r \qquad \xi \in \{\varepsilon, \mu, \alpha, \kappa\}.$$

From (7) we find the equations for the tangential components:

$$\frac{\mathrm{d}\boldsymbol{W}(r)}{\mathrm{d}r} = \mathrm{i}kM(r)\boldsymbol{W}(r) \tag{10}$$

where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \boldsymbol{W} = \begin{pmatrix} \boldsymbol{H}_\mathrm{t} \\ \boldsymbol{E}_\mathrm{t} \end{pmatrix}$$
$$A = \frac{\mathrm{i}}{kr}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi + \boldsymbol{e}_r^\times \alpha I + \boldsymbol{e}_r^\times \varepsilon \boldsymbol{e}_r \otimes \boldsymbol{v}_3 + \boldsymbol{e}_r^\times(\boldsymbol{u} + \alpha\boldsymbol{e}_r) \otimes \boldsymbol{v}_1$$
$$B = \boldsymbol{e}_r^\times \varepsilon I + \boldsymbol{e}_r^\times \varepsilon \boldsymbol{e}_r \otimes \boldsymbol{v}_4 + \boldsymbol{e}_r^\times(\boldsymbol{u} + \alpha\boldsymbol{e}_r) \otimes \boldsymbol{v}_2 \tag{11}$$
$$C = -\boldsymbol{e}_r^\times \mu I - \boldsymbol{e}_r^\times \mu \boldsymbol{e}_r \otimes \boldsymbol{v}_1 + \boldsymbol{e}_r^\times(\boldsymbol{u} - \kappa\boldsymbol{e}_r) \otimes \boldsymbol{v}_3$$
$$D = \frac{\mathrm{i}}{kr}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi - \boldsymbol{e}_r^\times \kappa I - \boldsymbol{e}_r^\times \mu \boldsymbol{e}_r \otimes \boldsymbol{v}_2 + \boldsymbol{e}_r^\times(\boldsymbol{u} - \kappa\boldsymbol{e}_r) \otimes \boldsymbol{v}_4.$$

The operator matrix M is the operator of the six-dimensional space $\mathcal{V} = \mathcal{U} \oplus \mathcal{U}$, blocks A, B, C, D are the planar tensors in the three-dimensional space (for a planar tensor A condition $A\boldsymbol{e}_r = \boldsymbol{e}_r A = 0$ holds true). At the same time projectors of the operator M determine its invariant subspaces, and two-dimensional subspace generated by vectors $\binom{\boldsymbol{e}_r}{0}$ and $\binom{0}{\boldsymbol{e}_r}$ is the proper subspace with zero eigenvalue. The operator matrix M can also be presented as a matrix in the four-dimensional subspace, then the blocks A, B, C, D are the 2×2 matrices.

Expressions (8)–(11) for a circular layer are written in a form analogous to well-known formulae for a stratified slab [10, 21] when $r \to \infty$. The fundamental solution of (10) is expressed by an evolution operator Ω_a^r of a circular layer (a, r):

$$\boldsymbol{W}(r) = \Omega_a^r[\mathrm{i}kM(r)]\boldsymbol{W}(a) \qquad a \neq 0. \tag{12}$$

Evolution operator is a product integral (matrizant) [18, 25] and has the form

$$\Omega_a^r[\mathrm{i}kM] = \int_a^r (E + \mathrm{i}kM(r)\,\mathrm{d}r) \qquad E = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From (10) one can obtain a tensor Riccati equation for an impedance tensor Γ [22]:

$$\frac{1}{\mathrm{i}k}\frac{\mathrm{d}\Gamma}{\mathrm{d}r} + \Gamma B\Gamma + \Gamma A - D\Gamma - C = 0. \tag{13}$$

The planar impedance tensor $\Gamma(r)$ relates the tangential components of the electric and magnetic field strength as $\boldsymbol{E}_\mathrm{t} = \Gamma\boldsymbol{H}_\mathrm{t}$. The independent solutions of (13) are two impedance tensors $\Gamma_1(r)$ and $\Gamma_2(r)$. In the layer $0 \leqslant r \leqslant a$ ($a \leqslant r \leqslant \infty$) one of the solutions has a peculiarity in the point $r = 0$ ($r = \infty$). Therefore, in these layers only one of the tensors $\Gamma_1(r)$, $\Gamma_2(r)$ is realized.

An impedance tensor of a homogeneous bianisotropic slab is constant. For a homogeneous circular layer an impedance tensor always depends on the radial coordinate r, because the planar tensors A, B, C, D always depend on r (even for homogeneous isotropic media). That is why we cannot simplify a tensor Riccati equation reducing it to an algebraic equation—a tensor quadratic equation. In [10], iterative methods of numerical solution of (13) are given.

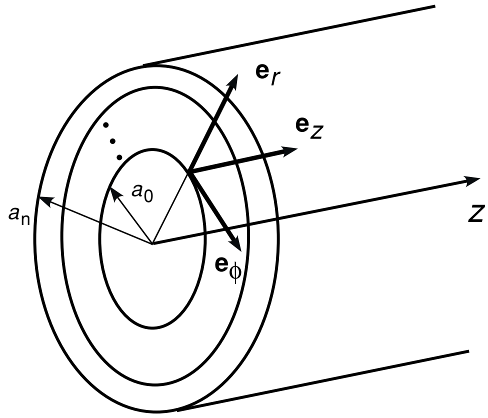

Figure 1. Geometry of a multilayer fibre.

3. Covariant dispersion equations of multilayer bianisotropic fibres

Let us consider a circular bianisotropic waveguide represented in figure 1. It consists of a core, a cladding and n intermediate layers with the following tensor electromagnetic characteristics of a medium:

$$(\varepsilon, \mu, \alpha, \kappa) = \begin{cases} (\varepsilon_{\rm co}, \mu_{\rm co}, \alpha_{\rm co}, \kappa_{\rm co}) & \text{for} \quad r < a_0 \\ (\varepsilon_j, \mu_j, \alpha_j, \kappa_j) & \text{for} \quad a_{j-1} \leqslant r < a_j \qquad j = 1, \ldots, n. \\ (\varepsilon_{\rm cl}, \mu_{\rm cl}, \alpha_{\rm cl}, \kappa_{\rm cl}) & \text{for} \quad r \geqslant a_n \end{cases} \tag{14}$$

If we solve equation (10) for each fibre layer, i.e. we find evolution operators of the intermediate layers and impedance tensors of the core and cladding, we will determine a dispersion equation of such a waveguide, its polarization and energy characteristics. In fact, the tangential components of the electric and magnetic fields on the boundaries between circular layers are related according to (12) as

$$\boldsymbol{W}^{(j)}(a_j) = \Omega_{a_{j-1}}^{a_j} \boldsymbol{W}^{(j)}(a_{j-1}) \qquad j = 1, \ldots, n$$

where $\Omega_{a_{j-1}}^{a_j}$ and $\boldsymbol{W}^{(j)}$ are the evolution operator and the tangential field components of the jth layer, respectively. The boundary conditions (continuity of the amplitudes $\boldsymbol{H}_{\rm t}$, $\boldsymbol{E}_{\rm t}$) are

$$\boldsymbol{W}^{(\rm co)}(a_0) = \boldsymbol{W}^{(1)}(a_0) \qquad \boldsymbol{W}^{(1)}(a_1) = \boldsymbol{W}^{(2)}(a_1) \quad \ldots \quad \boldsymbol{W}^{(n)}(a_n) = \boldsymbol{W}^{(\rm cl)}(a_n) \tag{15}$$

where $\boldsymbol{W}^{(\rm co)}$ and $\boldsymbol{W}^{(\rm cl)}$ are the tangential field components in the core and cladding. Then $\boldsymbol{W}^{(\rm co)}$ and $\boldsymbol{W}^{(\rm cl)}$ are connected by means of the evolution operator of all intermediate layers

$$\Omega_{a_0}^{a_n} = \Omega_{a_{n-1}}^{a_n} \Omega_{a_{n-2}}^{a_{n-1}} \ldots \Omega_{a_0}^{a_1}$$

by the relationship

$$\boldsymbol{W}^{(\rm cl)}(a_n) = \Omega_{a_0}^{a_n} \boldsymbol{W}^{(\rm co)}(a_0). \tag{16}$$

Using the surface impedance tensors of the core $\Gamma_{\rm co} = \Gamma_{\rm co}(a_0)$ and cladding $\Gamma_{\rm cl} = \Gamma_{\rm cl}(a_n)$ and multiplying (16) by the row block matrix $(\Gamma_{\rm cl} \;\; -I)$ we obtain

$$\Theta \boldsymbol{H}_{\rm t}^{(\rm co)}(a_0) = 0 \qquad \Theta = (\Gamma_{\rm cl} \;\; -I)\Omega_{a_0}^{a_n} \begin{pmatrix} I \\ \Gamma_{\rm co} \end{pmatrix} \tag{17}$$

where $\boldsymbol{H}_{\mathrm{t}}^{(\mathrm{co})}(a_0)$ are the tangential components of the magnetic field on the interface $r = a_0$. If there are no intermediate layers ($n = 0$), then $\Omega_{a_0}^{a_0} = E$ and $\Theta = \Gamma_{\mathrm{cl}} - \Gamma_{\mathrm{co}}$. In the three-dimensional space a planar tensor Θ has two eigenvalues equal to zero, which correspond to eigenvectors $\boldsymbol{e}_r$ and $\boldsymbol{H}_{\mathrm{t}}^{(\mathrm{co})}(a_0)$. Therefore, Θ is a dyad and expression

$$\mathrm{tr}(\overline{\Theta}) = 0 \tag{18}$$

determines a dispersion equation of a multilayer bianisotropic fibre. Invariant $\mathrm{tr}(\overline{\Theta})$ is the trace of an adjoint tensor (an adjoint tensor $\overline{\Theta}$ is defined by the expression $\overline{\Theta}\Theta = \Theta\overline{\Theta} = \det(\Theta)1$, where 1 is the identity tensor in the three-dimensional space) [14, 17]. Taking into consideration $\overline{\Theta} = \mathrm{tr}(\overline{\Theta})\boldsymbol{e}_r \otimes \boldsymbol{e}_r = 0$ and the Cayley–Hamilton theorem

$$\overline{\Theta} - \mathrm{tr}(\overline{\Theta})1 = \Theta\,(\Theta - \mathrm{tr}(\Theta)1) \tag{19}$$

we represent the dispersion equation (18) in the form [22]

$$\mathrm{tr}(\Theta^2) = (\mathrm{tr}(\Theta))^2. \tag{20}$$

The dispersion equation (20) can be expressed in terms of components of the planar tensor $\Theta = \Theta_{11}\boldsymbol{e}_z \otimes \boldsymbol{e}_z + \Theta_{12}\boldsymbol{e}_z \otimes \boldsymbol{e}_\varphi + \Theta_{21}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_z + \Theta_{22}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi$ (dispersion equation in the two-dimensional subspace is the determinant of 2×2 matrix Θ):

$$\Theta_{11}\Theta_{22} - \Theta_{12}\Theta_{21} = 0. \tag{21}$$

The dispersion equations (18), (20) and (21) are equivalent and can be written for any multilayer fibre, if we know the evolution operators and impedances of the corresponding layers. The solution of a dispersion equation determines a connection between the mode propagation constant β and the electromagnetic wave frequency ω. The modes of a circular fibre are expressed from the dispersion equation for different integer numbers ν and have certain polarizations. Let us find the tangential components of the magnetic field on the interface $r = a_0$. Using (17) and the Cayley–Hamilton theorem (19), we obtain

$$\boldsymbol{H}_{\mathrm{t}}^{(\mathrm{co})}(a_0) = (\Theta - \mathrm{tr}(\Theta)I)\boldsymbol{p} \tag{22}$$

where $\boldsymbol{p}$ is an arbitrary vector satisfying the condition $(\Theta - \mathrm{tr}(\Theta)I)\boldsymbol{p} \neq 0$. In the general case, a mode polarization can be characterized by the ratio of the electric and magnetic longitudinal field components [6]

$$\delta = \frac{E_z}{\mathrm{i}H_z} = \frac{\boldsymbol{e}_z\Gamma_{\mathrm{co}}(a)(\Theta - \mathrm{tr}(\Theta)I)\boldsymbol{p}}{\mathrm{i}\boldsymbol{e}_z(\Theta - \mathrm{tr}(\Theta)I)\boldsymbol{p}}. \tag{23}$$

The values $\delta > 0$ and $\delta < 0$ correspond to the hybrid HE- and EH-modes, respectively. If $\delta = 0$, then a mode is a transverse electrical one (TE); if $\delta = \infty$, then a mode is called a transverse magnetic mode (TM). TE- and TM-modes are realized in multilayer isotropic fibres at $\nu = 0$.

Energy flux in the direction of radiation propagation is the projection of the pointing vector onto the longitudinal axis z

$$S_z = \frac{c}{16\pi}\boldsymbol{W}^+(r)V^+\begin{pmatrix} 0 & \boldsymbol{e}_z^\times \\ -\boldsymbol{e}_z^\times & 0 \end{pmatrix} V\boldsymbol{W}(r)$$

where + is the Hermitian conjugation. By substituting operator (8) we obtain (the symbol $*$ denotes complex conjugation)

$$S_z = \frac{c}{16\pi}\boldsymbol{W}^+(r)\begin{pmatrix} \boldsymbol{v}_2^* \otimes \boldsymbol{e}_\varphi + \boldsymbol{e}_\varphi \otimes \boldsymbol{v}_3 & -\boldsymbol{v}_1^* \otimes \boldsymbol{e}_\varphi + \boldsymbol{e}_\varphi \otimes \boldsymbol{v}_4 \\ \boldsymbol{v}_4^* \otimes \boldsymbol{e}_\varphi - \boldsymbol{e}_\varphi \otimes \boldsymbol{v}_1 & -\boldsymbol{v}_3^* \otimes \boldsymbol{e}_\varphi - \boldsymbol{e}_\varphi \otimes \boldsymbol{v}_2 \end{pmatrix}\boldsymbol{W}(r). \tag{24}$$

The energy characteristic of a waveguide is the mode energy localization coefficient. It is equal to [6]

$$\rho = \frac{P_{\rm co}}{P} \qquad P_{\rm co} = \int_0^{a_0} S_z(r) r \, \mathrm{d}r \qquad P = \int_0^{\infty} S_z(r) r \, \mathrm{d}r. \tag{25}$$

Dispersion equation (18) in the analogous form has been considered earlier for multilayer planar waveguides (see [10, 22]). For a multilayer fibre (18) can be easily programmed for solving numerically.

4. Evolution operators and impedance tensors of homogeneous bianisotropic media

Let us solve equation (10) in the case of a homogeneous medium, when the components ε_{ij}, μ_{ij}, α_{ij}, κ_{ij} in (4) are constant. Then the operator matrix M can be presented as a power expansion of $1/r$:

$$M = M^{(0)} + \frac{1}{r} M^{(1)} + \frac{1}{r^2} M^{(2)} \tag{26}$$

where constant matrices $M^{(0)}$, $M^{(1)}$, $M^{(2)}$ can be easily found. We do not demonstrate them, because they are cumbersome. Further, we will determine the general structure of these matrices.

The six-dimensional space $\mathcal{V}$ can be represented by the direct product of the three-dimensional $\mathcal{U}$ and two-dimensional $\mathcal{W}$ spaces. $\mathcal{W}$ is generated by the unit vectors $\vec{e}_1 = \binom{1}{0}$, $\vec{e}_2 = \binom{0}{1}$. Therefore, $\boldsymbol{W}$ can be decomposed by means of the basis vectors of the space $\mathcal{W}$

$$\boldsymbol{W} = \boldsymbol{H}_{\rm t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boldsymbol{E}_{\rm t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{H}_{\rm t} \vec{e}_1 + \boldsymbol{E}_{\rm t} \vec{e}_2$$

or the space $\mathcal{U}$

$$\boldsymbol{W} = \vec{w}_\varphi \boldsymbol{e}_\varphi + \vec{w}_z \boldsymbol{e}_z \qquad \vec{w}_\varphi = \boldsymbol{e}_\varphi \boldsymbol{W} = \begin{pmatrix} \boldsymbol{e}_\varphi \boldsymbol{H} \\ \boldsymbol{e}_\varphi \boldsymbol{E} \end{pmatrix} = \begin{pmatrix} H_\varphi \\ E_\varphi \end{pmatrix} \qquad \vec{w}_z = \begin{pmatrix} H_z \\ E_z \end{pmatrix}.$$

The block matrix M also has two representations:

$$M = A \vec{e}_1 \otimes \vec{e}_1 + B \vec{e}_1 \otimes \vec{e}_2 + C \vec{e}_2 \otimes \vec{e}_1 + D \vec{e}_2 \otimes \vec{e}_2$$

and

$$M = M_{zz} \boldsymbol{e}_z \otimes \boldsymbol{e}_z + M_{z\varphi} \boldsymbol{e}_z \otimes \boldsymbol{e}_\varphi + M_{\varphi z} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_z + M_{\varphi\varphi} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi$$
$$M_{zz} = \boldsymbol{e}_z M \boldsymbol{e}_z = \begin{pmatrix} \boldsymbol{e}_z A \boldsymbol{e}_z & \boldsymbol{e}_z B \boldsymbol{e}_z \\ \boldsymbol{e}_z C \boldsymbol{e}_z & \boldsymbol{e}_z D \boldsymbol{e}_z \end{pmatrix} = \begin{pmatrix} A_{zz} & B_{zz} \\ C_{zz} & D_{zz} \end{pmatrix}$$
$$M_{z\varphi} = \boldsymbol{e}_z M \boldsymbol{e}_\varphi \qquad M_{\varphi z} = \boldsymbol{e}_\varphi M \boldsymbol{e}_z \qquad M_{\varphi\varphi} = \boldsymbol{e}_\varphi M \boldsymbol{e}_\varphi.$$

The notation of the operator matrix M in terms of the planar tensors A, B, C, D has already been used in (11). M_{zz}, $M_{z\varphi}$, $M_{\varphi z}$ and $M_{\varphi\varphi}$ are the matrices of the space $\mathcal{W}$.

One can find the solution of equation (10) in several ways. The first way is to solve the differential equation of the first order (10) for the vector $\boldsymbol{W}$ directly. In that case the solution can be expressed by the power series of r. The second method is based on solving a differential equation of second order for the tangential components of the magnetic $\boldsymbol{H}_{\rm t}$ or electric $\boldsymbol{E}_{\rm t}$ field. We will solve equation (10) in a third way, i.e. solve a differential equation of second order for $\vec{w}_z$. Then for bianisotropic media matrices $M^{(0)}$, $M^{(1)}$, $M^{(2)}$ take the following form:

$$\begin{aligned} M^{(0)} &= M_{zz}^{(0)} \boldsymbol{e}_z \otimes \boldsymbol{e}_z + M_{z\varphi}^{(0)} \boldsymbol{e}_z \otimes \boldsymbol{e}_\varphi + M_{\varphi z}^{(0)} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_z + M_{\varphi\varphi}^{(0)} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \\ M^{(1)} &= M_{zz}^{(1)} \boldsymbol{e}_z \otimes \boldsymbol{e}_z + M_{\varphi z}^{(1)} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_z + M_{\varphi\varphi}^{(1)} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \qquad M^{(2)} = M_{\varphi z}^{(2)} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_z. \end{aligned} \tag{27}$$

In this representation some matrices become zero: $M^{(1)}_{z\varphi} = M^{(2)}_{zz} = M^{(2)}_{z\varphi} = M^{(2)}_{\varphi\varphi} = 0$. Therefore, $M_{z\varphi} = M^{(0)}_{z\varphi}$ is a constant matrix.

From (10) we obtain the differential equation for $\vec{w}_z$:

$$\vec{w}''_z + P\vec{w}'_z + Q\vec{w}_z = 0 \tag{28}$$

where

$$P = P^{(0)} + \frac{1}{r}P^{(1)} \qquad Q = Q^{(0)} + \frac{1}{r}Q^{(1)} + \frac{1}{r^2}Q^{(2)}. \tag{29}$$

Constant matrices $P^{(0)}$, $P^{(1)}$, $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$ of the space $\mathcal{W}$ equal

$$\begin{aligned}
&P^{(0)} = -\mathrm{i}k\big(M^{(0)}_{zz} + M^{(0)}_{z\varphi}M^{(0)}_{\varphi\varphi}M^{(0)-1}_{z\varphi}\big) \qquad P^{(1)} = -\mathrm{i}k\big(M^{(1)}_{zz} + M^{(0)}_{z\varphi}M^{(1)}_{\varphi\varphi}M^{(0)-1}_{z\varphi}\big)\\
&Q^{(0)} = k^2\big(M^{(0)}_{z\varphi}M^{(0)}_{\varphi z} - M^{(0)}_{z\varphi}M^{(0)}_{\varphi\varphi}M^{(0)-1}_{z\varphi}M^{(0)}_{zz}\big)\\
&Q^{(1)} = k^2\big(M^{(0)}_{z\varphi}M^{(1)}_{\varphi z} - M^{(0)}_{z\varphi}M^{(1)}_{\varphi\varphi}M^{(0)-1}_{z\varphi}M^{(0)}_{zz} - M^{(0)}_{z\varphi}M^{(0)}_{\varphi\varphi}M^{(0)-1}_{z\varphi}M^{(1)}_{zz}\big)\\
&Q^{(2)} = \mathrm{i}kM^{(1)}_{zz} + k^2M^{(0)}_{z\varphi}M^{(2)}_{\varphi z} - k^2M^{(0)}_{z\varphi}M^{(1)}_{\varphi\varphi}M^{(0)-1}_{z\varphi}M^{(1)}_{zz}.
\end{aligned} \tag{30}$$

According to [26] equation (28), which can be reduced to the system of four ordinary differential equations of the first order, has a fundamental sequence of four regular solutions near $r = 0$. Therefore, we search the solution of equation (28) in the form of a power series (Frobenius method)

$$\vec{w}_z = \sum_{l=0}^{\infty} r^{\sigma+l}\vec{w}^{(l)}_z. \tag{31}$$

Substitution of (31) in (28) gives the relationship

$$\Delta_l\vec{w}^{(l)}_z + \Upsilon_l\vec{w}^{(l-1)}_z + Q^{(0)}\vec{w}^{(l-2)}_z = 0 \tag{32}$$

where ($\hat{1}$ is the identity tensor in the two-dimensional space $\mathcal{W}$)

$$\Delta_l = (l+\sigma)(l+\sigma-1)\hat{1} + (l+\sigma)P^{(1)} + Q^{(2)} \qquad \Upsilon_l = (l+\sigma-1)P^{(0)} + Q^{(1)}. \tag{33}$$

At $l = 0$ we obtain

$$\Delta_0\vec{w}^{(0)}_z = 0 \tag{34}$$

from which we can write a condition for σ

$$\det(\Delta_0) \equiv \det(\sigma(\sigma-1)\hat{1} + \sigma P^{(1)} + Q^{(2)}) = 0 \tag{35}$$

and an expression for the vector $\vec{w}^{(0)}_z$

$$\vec{w}^{(0)}_z = (\Delta_0 - \mathrm{tr}(\Delta_0)\hat{1})c\vec{a} \tag{36}$$

where c is a scalar constant, an arbitrary vector $\vec{a}$ satisfies $(\Delta_0 - \mathrm{tr}(\Delta_0)\hat{1})\vec{a} \neq 0$. Equation (35) gives four roots $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. If the difference between any two roots is not an integer number, then each of four fundamental solutions can be presented as a power series (31). The vector $\vec{w}^{(0)}_z$ determines a one-dimensional space of solutions: one constant c_j corresponds to one value σ_j, $j = 1, 2, 3, 4$, and (36) holds true. Otherwise the fundamental solution for the greater root is expressed by the power series and for other roots can contain logarithms [26]. If $\Delta_0 = 0$, then the space of solutions is two dimensional. There are two couples of roots of equation (35) (for example, $\sigma_1 = \sigma_2, \sigma_3 = \sigma_4$), and an arbitrary vector $\vec{w}^{(0)}_z$ determines a couple of constants: $\vec{w}^{(0)}_z = c_1\vec{e}_1 + c_2\vec{e}_2$ or $\vec{w}^{(0)}_z = c_3\vec{e}_1 + c_4\vec{e}_2$.

At $l = 1$ we find $\vec{w}^{(1)}_z$:

$$\vec{w}^{(1)}_z = -\Delta_1^{-1}\Upsilon_1\vec{w}^{(0)}_z. \tag{37}$$

Further, we can write the following recurrence relation, which expresses $\vec{w}_z^{(l)}$ in terms of $\vec{w}_z^{(l-1)}$ and $\vec{w}_z^{(l-2)}$:

$$\vec{w}_z^{(l)} = -\Delta_l^{-1}\big(\Upsilon_l \vec{w}_z^{(l-1)} + Q^{(0)}\vec{w}_z^{(l-2)}\big). \tag{38}$$

We write the general solution of equation (28) as a linear combination of four independent solutions with constant coefficients c_j:

$$\vec{w}_z(r) = \sum_{j=1}^{4} T_j(r) c_j \vec{a}_j \tag{39}$$

where $T_j(r)$ is an independent solution expressed by the 2×2 matrix and $\vec{a}_j$ is an arbitrary vector defined in (36).

Using the known solution (39), let us find an evolution operator Ω_a^r and an impedance tensor Γ. The vector $\boldsymbol{W}$ equals

$$\boldsymbol{W} = \sum_{j=1}^{4}(T_j(r)\vec{a}_j \boldsymbol{e}_z + \hat{Z}T_j\vec{a}_j\boldsymbol{e}_\varphi)c_j \tag{40}$$

where the differential operator $\hat{Z}$ is expressed by the formula

$$\hat{Z} = M_{z\varphi}^{-1}\left(\frac{1}{\mathrm{i}k}\frac{\mathrm{d}}{\mathrm{d}r} - M_{zz}\right).$$

Considering constants c_j as the components of vectors $\boldsymbol{c}_1 = c_1\boldsymbol{e}_z + c_2\boldsymbol{e}_\varphi$ and $\boldsymbol{c}_2 = c_3\boldsymbol{e}_z + c_4\boldsymbol{e}_\varphi$ of the space $\mathcal{U}$, the vector $\boldsymbol{W}$ can be written in the form

$$\boldsymbol{W} = S(r)\boldsymbol{C} \qquad S = \begin{pmatrix}\eta_1 & \eta_2\\ \zeta_1 & \zeta_2\end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix}\boldsymbol{c}_1\\ \boldsymbol{c}_2\end{pmatrix}. \tag{41}$$

The blocks of the matrix S are planar tensors of the three-dimensional space $\mathcal{U}$. They are equal to

$$\begin{aligned}
\eta_1 &= \vec{e}_1 T_1\vec{a}_1\boldsymbol{e}_z\otimes\boldsymbol{e}_z + \vec{e}_1\hat{Z}T_1\vec{a}_1\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_z + \vec{e}_1 T_2\vec{a}_2\boldsymbol{e}_z\otimes\boldsymbol{e}_\varphi + \vec{e}_1\hat{Z}T_2\vec{a}_2\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\varphi\\
\eta_2 &= \vec{e}_1 T_3\vec{a}_3\boldsymbol{e}_z\otimes\boldsymbol{e}_z + \vec{e}_1\hat{Z}T_3\vec{a}_3\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_z + \vec{e}_1 T_4\vec{a}_4\boldsymbol{e}_z\otimes\boldsymbol{e}_\varphi + \vec{e}_1\hat{Z}T_4\vec{a}_4\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\varphi\\
\zeta_1 &= \vec{e}_2 T_1\vec{a}_1\boldsymbol{e}_z\otimes\boldsymbol{e}_z + \vec{e}_2\hat{Z}T_1\vec{a}_1\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_z + \vec{e}_2 T_2\vec{a}_2\boldsymbol{e}_z\otimes\boldsymbol{e}_\varphi + \vec{e}_2\hat{Z}T_2\vec{a}_2\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\varphi\\
\zeta_2 &= \vec{e}_2 T_3\vec{a}_3\boldsymbol{e}_z\otimes\boldsymbol{e}_z + \vec{e}_2\hat{Z}T_3\vec{a}_3\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_z + \vec{e}_2 T_4\vec{a}_4\boldsymbol{e}_z\otimes\boldsymbol{e}_\varphi + \vec{e}_2\hat{Z}T_4\vec{a}_4\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\varphi.
\end{aligned} \tag{42}$$

The evolution operator Ω_a^r, which relates the tangential components in the points a and r, is determined by the relationship

$$\Omega_a^r = S(r)S^-(a) \tag{43}$$

where S^- is a pseudoinverse matrix ($SS^- = S^-S = E$) [14, 17]. Expression (41) can be easily represented as a superposition of two waves

$$\boldsymbol{W} = \begin{pmatrix}\boldsymbol{H}_{\mathrm{t}1}\\ \boldsymbol{E}_{\mathrm{t}1}\end{pmatrix} + \begin{pmatrix}\boldsymbol{H}_{\mathrm{t}2}\\ \boldsymbol{E}_{\mathrm{t}2}\end{pmatrix} \tag{44}$$

where $\boldsymbol{H}_{\mathrm{t}1} = \eta_1\boldsymbol{c}_1, \boldsymbol{H}_{\mathrm{t}2} = \eta_2\boldsymbol{c}_2, \boldsymbol{E}_{\mathrm{t}1} = \zeta_1\boldsymbol{c}_1$ and $\boldsymbol{E}_{\mathrm{t}2} = \zeta_2\boldsymbol{c}_2$. Using the definition of an impedance tensor $\boldsymbol{E}_{\mathrm{t}m} = \Gamma_m\boldsymbol{H}_{\mathrm{t}m}$ we obtain

$$(\zeta_m - \Gamma_m\eta_m)\,\boldsymbol{c}_m = 0 \qquad m = 1, 2. \tag{45}$$

Since (45) should hold for any vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ the impedance tensors of each wave equal

$$\Gamma_m = \zeta_m\eta_m^- \tag{46}$$

where η_m^- is a pseudoinverse tensor ($\eta_m^-\eta_m = \eta_m\eta_m^- = I$). If we know the initial wave amplitudes $\boldsymbol{H}_{t1}(a)$ and $\boldsymbol{H}_{t2}(a)$, then the field evolution is expressed as

$$\boldsymbol{H}_{tm}(r) = \eta_m(r)\eta_m^-(a)\boldsymbol{H}_{tm}(a) \qquad \boldsymbol{E}_{tm}(r) = \zeta_m(r)\eta_m^-(a)\boldsymbol{H}_{tm}(a). \tag{47}$$

Thus, we have found the general expressions for the evolution operator (43) and the impedance tensor (46) of a circular bianisotropic layer. These formulae are written for the known solutions T_j, $j = 1, 2, 3, 4$ and used for an analysis of dispersion equations. In the next section we obtain some particular solutions T_j and fibre dispersion equations.

5. Bessel solutions

Bianisotropic media allow us to realize a number of particular cases of solving equation (28). The solution is determined by the five tensors $P^{(0)}$, $P^{(1)}$, $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$. In this section, we investigate the connection between the form of these tensors and the type of a bianisotropic medium.

Assuming $P^{(0)} = 0$, $Q^{(1)} = 0$ we obtain the equation (it is similar to the Bessel equation, but with tensor coefficients):

$$\vec{w}_z'' + \frac{1}{r}P^{(1)}\vec{w}_z' + \left(Q^{(0)} + \frac{1}{r^2}Q^{(2)}\right)\vec{w}_z = 0. \tag{48}$$

In that case $\Upsilon_l = 0$, and therefore $\vec{w}_z^{(2p+1)} = 0$, $p = 0, 1, 2, \ldots$. Non-zero terms of the series (31) are the even terms:

$$\vec{w}_z^{(2p)} = N^{(p)}\vec{w}_z^{(0)} \qquad N^{(0)} = \hat{1} \qquad N^{(p)} = (-1)^p\Delta_{2p}^{-1}Q^{(0)}\Delta_{2p-2}^{-1}Q^{(0)}\ldots\Delta_2^{-1}Q^{(0)} \tag{49}$$

wherefrom solutions (39) are equal to

$$T_j = \sum_{p=0}^{\infty} r^{2p+\sigma_j}N^{(p)}. \tag{50}$$

Let us add conditions $P^{(1)} = \hat{1}$, $Q^{(2)} = -b\nu^2\hat{1}$ to the relationships $P^{(0)} = 0$, $Q^{(1)} = 0$ (b is a constant). The only arbitrary tensor is $Q^{(0)}$. Equation (28) takes the form

$$\vec{w}_z'' + \frac{1}{r}\vec{w}_z' + \left(Q^{(0)} - \frac{b\nu^2}{r^2}\hat{1}\right)\vec{w}_z = 0. \tag{51}$$

Then $\Delta_l = ((l+\sigma)^2 - b\nu^2)\hat{1}$ is proportional to the identity tensor of the space $\mathcal{W}$. Equation (35) gives two couples of the same roots $\sigma_1 = \sigma_2 = \sqrt{b}\nu$, $\sigma_3 = \sigma_4 = -\sqrt{b}\nu$. At integer values of $\sqrt{b}\nu$ (for example, $b = 1$) the solutions can be written in terms of the Bessel functions [27] with the tensor argument $\sqrt{Q^{(0)}}r$

$$T_1 = T_2 = J_{\sqrt{b}\nu}\big(\sqrt{Q^{(0)}}r\big) \qquad T_3 = T_4 = Y_{\sqrt{b}\nu}\big(\sqrt{Q^{(0)}}r\big)$$

or the modified Bessel functions

$$T_1 = T_2 = I_{\sqrt{b}\nu}\big(\sqrt{-Q^{(0)}}r\big) \qquad T_3 = T_4 = K_{\sqrt{b}\nu}\big(\sqrt{-Q^{(0)}}r\big).$$

At fractional values of $\sqrt{b}\nu$ we have the following linearly independent solutions:

$$T_1 = T_2 = J_{\sqrt{b}\nu}\big(\sqrt{Q^{(0)}}r\big) \qquad T_3 = T_4 = J_{-\sqrt{b}\nu}\big(\sqrt{Q^{(0)}}r\big).$$

The vectors $\vec{a}_j$ equal $\vec{a}_1 = \vec{a}_3 = \vec{e}_1$, $\vec{a}_2 = \vec{a}_4 = \vec{e}_2$. Tensor $Q^{(0)}$ can be written in the spectral form: $Q^{(0)} = \lambda_1\tau_1 + \lambda_2\tau_2$, where λ_1, λ_2 are the eigenvalues of the tensor $Q^{(0)}$,

Table 1. Examples of bianisotropic media for some values of tensors $P^{(0)}$, $P^{(1)}$, $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$ (χ is the parameter which can be called the susceptibility).

Tensors $P^{(0)}$, $P^{(1)}$, $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$	Bianisotropic medium, $\xi \in \{\varepsilon, \mu, \alpha, \kappa\}$
$P^{(0)} = 0$, the rest tensors are arbitrary	(a) $\xi = \xi_{rr} e_r \otimes e_r + \xi_{\varphi\varphi} e_\varphi \otimes e_\varphi + \xi_{zz} e_z \otimes e_z + \xi_{\varphi r} e_\varphi \otimes e_r + \xi_{r\varphi} e_r \otimes e_\varphi$ (b) $\varepsilon = \varepsilon_{rr} e_r \otimes e_r + \varepsilon_{\varphi\varphi} e_\varphi \otimes e_\varphi + \varepsilon_{zz} e_z \otimes e_z + \varepsilon_{rz} e_\varphi^\times$ $\mu = \mu_{rr} e_r \otimes e_r + \mu_{\varphi\varphi} e_\varphi \otimes e_\varphi + \mu_{zz} e_z \otimes e_z + \mu_{rz} e_\varphi^\times$ $\alpha = \alpha_{zz} e_z \otimes e_z \qquad \kappa = \kappa_{zz} e_z \otimes e_z$
$P^{(0)} = 0$, $Q^{(1)} = 0$, the rest tensors are arbitrary	$\xi = \xi_{rr} e_r \otimes e_r + \xi_{\varphi\varphi} e_\varphi \otimes e_\varphi + \xi_{zz} e_z \otimes e_z + \xi_{\varphi r} e_\varphi \otimes e_r + \xi_{r\varphi} e_r \otimes e_\varphi$
$P^{(0)} = 0$, $Q^{(1)} = 0$ $P^{(1)} = \hat{1}$, $Q^{(2)} = -b\nu^2 \hat{1}$ $Q^{(0)}$ is arbitrary	(a) $\xi = \xi_{rr} e_r \otimes e_r + \xi_{\varphi\varphi} e_\varphi \otimes e_\varphi + \xi_{zz} e_z \otimes e_z$ $\varepsilon_{\varphi\varphi}\mu_{rr} = \varepsilon_{rr}\mu_{\varphi\varphi} \qquad \varepsilon_{rr}\alpha_{\varphi\varphi} = \varepsilon_{\varphi\varphi}\kappa_{rr}$ $\varepsilon_{rr}\kappa_{\varphi\varphi} = \alpha_{rr}\varepsilon_{\varphi\varphi} \qquad b = \frac{\mu_{\varphi\varphi}}{\mu_{rr}} = \frac{\varepsilon_{\varphi\varphi}}{\varepsilon_{rr}}$ (b) $\xi = \xi_{zz} e_z \otimes e_z + \xi_{rr}(1 - e_z \otimes e_z) + \mathrm{i}\chi_\xi e_z^\times$

τ_1, τ_2 are the projection operators $\left(\tau_1\tau_2 = \tau_2\tau_1 = 0, \tau_1^2 = \tau_1, \tau_2^2 = \tau_2, \tau_1 + \tau_2 = \hat{1}\right)$. For $\lambda_1 \neq \lambda_2$ the projection operators are computed from the formulae

$$\tau_1 = \frac{Q^{(0)} - \lambda_2 \hat{1}}{\lambda_1 - \lambda_2} \qquad \tau_2 = -\frac{Q^{(0)} - \lambda_1 \hat{1}}{\lambda_1 - \lambda_2}. \tag{52}$$

Then $\sqrt{Q^{(0)}} = \sqrt{\lambda_1}\tau_1 + \sqrt{\lambda_2}\tau_2$ and the Bessel function is equal to

$$J_{\sqrt{b}\nu}(\sqrt{Q^{(0)}}r) = J_{\sqrt{b}\nu}(\sqrt{\lambda_1}r)\tau_1 + J_{\sqrt{b}\nu}(\sqrt{\lambda_2}r)\tau_2. \tag{53}$$

If $\lambda_1 = \lambda_2$ then $Q^{(0)} = \lambda_1(\tau_1 + \tau_2) = \lambda_1 \hat{1}$ and $J_{\sqrt{b}\nu}(\sqrt{Q^{(0)}}r) = J_{\sqrt{b}\nu}(\sqrt{\lambda_1}r)\hat{1}$.

In table 1, the correspondence between the tensors $P^{(0)}$, $P^{(1)}$, $Q^{(0)}$, $Q^{(1)}$, $Q^{(2)}$ and the type of bianisotropic medium is given. We see, that condition $P^{(0)} = 0$, $Q^{(1)} = 0$ holds true for a bianisotropic medium with different diagonal components of the tensors $\xi \in \{\varepsilon, \mu, \alpha, \kappa\}$ and arbitrary components $\xi_{r\varphi}, \xi_{\varphi r}$. Such dielectric permittivity tensor, magnetic permeability tensor and gyration pseudotensors have both symmetric and antisymmetric parts: $\xi = \xi_s + \xi_a$, where $\xi_s = \xi_{rr} e_r \otimes e_r + \xi_{\varphi\varphi} e_\varphi \otimes e_\varphi + \xi_{zz} e_z \otimes e_z + \frac{1}{2}(\xi_{r\varphi} + \xi_{\varphi r})(e_\varphi \otimes e_r + e_r \otimes e_\varphi)$, $\xi_a = \frac{1}{2}(\xi_{r\varphi} - \xi_{\varphi r}) e_z^\times$. Tensor ξ is symmetric, when $\xi_{r\varphi} = \xi_{\varphi r}$. In the general case bianisotropic media under consideration are absorbing. For non-absorbing media

$$\varepsilon^+ = \varepsilon, \qquad \mu^+ = \mu, \qquad \alpha^+ = \kappa. \tag{54}$$

$P^{(0)} = 0$ holds true for other bianisotropic media, the example of which being the medium (b) in table 1. Such a medium is non-absorbing, if ε_{rz}, μ_{rz} are imaginary, diagonal components of tensors ε, μ are real and $\alpha_{zz}^* = \kappa_{zz}$.

Condition $P^{(0)} = 0$, $Q^{(1)} = 0$, $P^{(1)} = \hat{1}$, $Q^{(2)} = -b\nu^2\hat{1}$ is realized for two types of bianisotropic media. In the first case the tensors $\xi \in \{\varepsilon, \mu, \alpha, \kappa\}$ are diagonal. Diagonal components satisfy the relationships given in table 1. The solutions of (28) are the Bessel functions of the fractional order $\pm\sqrt{b}\nu$. The second class of media corresponds to media isotropic in the fibre cross-section. This is the most important and practically applied case, which includes isotropic, anisotropic, gyrotropic, biisotropic, bianisotropic media. At the same time we can easily find solutions (39) (they are expressed by means of the Bessel functions), an evolution operator (43) of a circular layer, an impedance tensor (46), a fibre dispersion equation (18).

Let us consider a number of examples with $P^{(0)} = 0$, $Q^{(1)} = 0$, $P^{(1)} = \hat{1}$, $Q^{(2)} = -\nu^2\hat{1}$.

5.1. Isotropic medium

The simplest case is the case of a homogeneous isotropic medium, which is characterized by the scalar dielectric permittivity ε and magnetic permeability μ ($\alpha = 0$, $\kappa = 0$). Then

$$Q^{(0)} = (k^2\varepsilon\mu - \beta^2)\hat{1} \qquad T_1 = T_2 = F^{(1)}_{\nu\pm}(u_\pm r/a)\hat{1} \qquad T_3 = T_4 = F^{(2)}_{\nu\pm}(u_\pm r/a)\hat{1}$$

where $u^2_\pm = \pm k^2a^2(\varepsilon\mu - \beta^2/k^2)$, $r = a$ is a point of a circular layer, $F^{(1)}_{\nu+} = J_\nu(u_+r/a)$ is the Bessel function of the first kind, $F^{(2)}_{\nu+} = Y_\nu(u_+r/a)$ is the Bessel function of the second kind, $F^{(1)}_{\nu-} = I_\nu(u_-r/a)$ and $F^{(2)}_{\nu-} = K_\nu(u_-r/a)$ are the modified Bessel functions. $F^{(1)}$ and $F^{(2)}$ are the linearly independent solutions of the Bessel equation (one can choose any couple of them), the signs + and − correspond to the function arguments u_+ and u_-, respectively.

Tensors η_1, η_2, ζ_1, ζ_2 equal

$$\begin{aligned}\eta_m &= F^{(m)}_{\nu\pm}\left(e_z \mp \frac{\beta\nu a^2}{u^2_\pm r}e_\varphi\right)\otimes e_z \pm \frac{\mathrm{i}ka\varepsilon}{u_\pm}F^{(m)\prime}_{\nu\pm}e_\varphi\otimes e_\varphi\\ \zeta_m &= \mp\frac{\mathrm{i}ka\mu}{u_\pm}F^{(m)\prime}_{\nu\pm}e_\varphi\otimes e_z + F^{(m)}_{\nu\pm}\left(e_z \mp \frac{\beta\nu a^2}{u^2_\pm r}e_\varphi\right)\otimes e_\varphi\end{aligned} \tag{55}$$

where $F^{(m)\prime}_{\nu\pm}(x) = \mathrm{d}F^{(m)}_{\nu\pm}(x)/\mathrm{d}x$, $m = 1, 2$. The evolution operator and the impedance tensors are computed from formulae (43) and (46). For a waveguide with the dielectric permittivity and the magnetic permeability

$$\varepsilon = \begin{cases}\varepsilon_{\mathrm{co}} & \text{for} \quad r < a\\ \varepsilon_{\mathrm{cl}} & \text{for} \quad r \geqslant a\end{cases} \qquad \mu = \begin{cases}\mu_{\mathrm{co}} & \text{for} \quad r < a\\ \mu_{\mathrm{cl}} & \text{for} \quad r \geqslant a\end{cases}$$

one usually chooses solutions

$$F_\nu = \begin{cases}F^{(1)}_{\nu+} = J_\nu(u_+r/a) & \text{for} \quad r < a\\ F^{(2)}_{\nu-} = K_\nu(u_-r/a) & \text{for} \quad r \geqslant a.\end{cases}$$

The choice of solutions is caused by the fact that only Bessel function J_ν is finite at $r = 0$ and only K_ν tends to zero at $r = \infty$. Then the surface impedance tensors at the interface $r = a$ are of the form

$$\begin{aligned}\Gamma_{\mathrm{co}} &= -\frac{\mathrm{i}u}{ka\varepsilon_{\mathrm{co}}}\frac{J_\nu(u)}{J'_\nu(u)}\left(e_z - \frac{\beta\nu a}{u^2}e_\varphi\right)\otimes\left(e_\varphi + \frac{\beta\nu a}{u^2}e_z\right) - \frac{\mathrm{i}ka\mu_{\mathrm{co}}}{u}\frac{J'_\nu(u)}{J_\nu(u)}e_\varphi\otimes e_z\\ \Gamma_{\mathrm{cl}} &= \frac{\mathrm{i}w}{ka\varepsilon_{\mathrm{cl}}}\frac{K_\nu(w)}{K'_\nu(w)}\left(e_z + \frac{\beta\nu a}{w^2}e_\varphi\right)\otimes\left(e_\varphi - \frac{\beta\nu a}{w^2}e_z\right) + \frac{\mathrm{i}ka\mu_{\mathrm{cl}}}{w}\frac{K'_\nu(w)}{K_\nu(w)}e_\varphi\otimes e_z\end{aligned} \tag{56}$$

where $u^2 = k^2a^2(\varepsilon_{\mathrm{co}}\mu_{\mathrm{co}} - \beta^2/k^2)$, $w^2 = k^2a^2(\beta^2/k^2 - \varepsilon_{\mathrm{cl}}\mu_{\mathrm{cl}})$.

By substituting the surface impedance tensors into the dispersion equation (18) ($n = 0$), we obtain the well-known dispersion relation which one can find, for instance, in [6–8]. In the case of a multilayer isotropic fibre with n intermediate circular layers, the dispersion equation (18) includes the evolution operators of these layers

$$\Omega^{a_n}_{a_0} = S_n(a_n)S^-_n(a_{n-1})S_{n-1}(a_{n-1})S^-_{n-1}(a_{n-2})\ldots S_1(a_1)S^-_1(a_0). \tag{57}$$

5.2. Bianisotropic medium

Let us consider a fibre with a bianisotropic core $\varepsilon = \varepsilon_1(1 - \boldsymbol{e}_z \otimes \boldsymbol{e}_z) + \varepsilon_2 \boldsymbol{e}_z \otimes \boldsymbol{e}_z$, $\mu = \mu_1(1 - \boldsymbol{e}_z \otimes \boldsymbol{e}_z) + \mu_2 \boldsymbol{e}_z \otimes \boldsymbol{e}_z$, $\alpha = \kappa = \mathrm{i}\chi \boldsymbol{e}_z^\times$ and an isotropic cladding $\varepsilon_{\mathrm{cl}}$, μ_{cl}. In that case T_1, T_2 in the core are equal to

$$T_1 = T_2 = J_\nu(\sqrt{\mu_2/\mu_1}ur/a)\vec{e}_1 \otimes \vec{e}_1 + J_\nu(\sqrt{\varepsilon_2/\varepsilon_1}ur/a)\vec{e}_2 \otimes \vec{e}_2$$

where $u^2 = k^2a^2(\varepsilon_1\mu_1 - \chi^2 - \beta^2/k^2)$. Further, we find the surface impedance tensor of the core at the fibre interface $r = a$

$$\begin{aligned}\Gamma_{\mathrm{co}} = &-\frac{\mathrm{i}u}{ka\sqrt{\varepsilon_1\varepsilon_2}}\frac{J_\nu(\sqrt{\varepsilon_2/\varepsilon_1}u)}{J_\nu'(\sqrt{\varepsilon_2/\varepsilon_1}u)}\left(\boldsymbol{e}_z - \frac{(\beta + \mathrm{i}k\chi)\nu a}{u^2}\boldsymbol{e}_\varphi\right) \otimes \left(\boldsymbol{e}_\varphi + \frac{(\beta - \mathrm{i}k\chi)\nu a}{u^2}\boldsymbol{e}_z\right)\\ &- \frac{\mathrm{i}ka\sqrt{\mu_1\mu_2}}{u}\frac{J_\nu'(\sqrt{\mu_2/\mu_1}u)}{J_\nu(\sqrt{\mu_2/\mu_1}u)}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_z \end{aligned} \tag{58}$$

and the dispersion equation of the waveguide

$$\begin{aligned}&\left(\frac{J_\nu'(\sqrt{\mu_2/\mu_1}u)}{uJ_\nu(\sqrt{\mu_2/\mu_1}u)} + \frac{\mu_{\mathrm{cl}}}{\sqrt{\mu_1\mu_2}}\frac{K_\nu'(w)}{wK_\nu(w)}\right)\left(\frac{J_\nu'(\sqrt{\varepsilon_2/\varepsilon_1}u)}{uJ_\nu(\sqrt{\varepsilon_2/\varepsilon_1}u)} + \frac{\varepsilon_{\mathrm{cl}}}{\sqrt{\varepsilon_1\varepsilon_2}}\frac{K_\nu'(w)}{wK_\nu(w)}\right)\\ &\quad = \frac{\nu^2}{\sqrt{\varepsilon_1\varepsilon_2}\sqrt{\mu_1\mu_2}}\frac{\beta^2(u^2 + w^2)^2 + k^2w^4\chi^2}{k^2u^4w^4}.\end{aligned} \tag{59}$$

Dispersion equations of two-layer fibres are determined by the impedance tensors of a core and a cladding. The core impedance tensor is expressed in terms of the Bessel functions J_ν, the cladding impedance tensor in terms of the modified Bessel functions K_ν. At $r \to \infty$ the impedance tensors of a circular layer become the impedance tensors of a planar film.

6. Conclusion

Thus, the general matrix method for determining electromagnetic field in a circular bianisotropic layer is developed. We use it to obtain the evolution operators and the impedance tensors and to write the dispersion equations of multilayer fibres. As an example, we represent the dispersion relations of the fibres with isotropic and bianisotropic cores. More complex cases of multilayer fibres lead to cumbersome analytical results. That is why it is better to solve the dispersion equation (18) of a multilayer fibre using a computer. The dispersion equation can be easily programmed as an invariant of the multiplication of known block matrices. The mode polarization and the energy flux are expressed by formulae (23) and (24).

Proposed method has some advantages in comparison with existing approaches for getting dispersion relations. As one of the advantages, one can call on the generality of the technique which allows us to obtain the guided modes for a wide set of bianisotropic fibres. Furthermore, a dispersion equation is written in terms of the block matrices. The matrix computation (addition and multiplication) can be easily done giving the analytical formulae and numerical solutions of the problem. The simplicity of the proposed method in comparison with existing techniques lies in the clear algorithm to obtain a dispersion equation for an arbitrary multilayer fibre.

For a homogeneous circular bianisotropic layer, we reduce the 6×6 matrix equation (10) to the differential equation of the second order (28) with 2×2 matrix coefficients for the longitudinal field components. At some matrix coefficients, the solution of this equation is expressed by means of the Bessel functions with matrix (tensor) argument (it is realized for the media isotropic in the fibre cross-section).

The matrix method can be applied for analysing the propagation of the Bessel beams in complex media. Solutions obtained in the fibre core also describe the Bessel beams, the longitudinal component of the beam energy flux being expressed by (24).

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